10.5 IF ESTIMATION OF FM SIGNALS IN MULTIPLICATIVE NOISE [0]

10.5.1 Random Amplitude Modulation

Most IF estimation techniques, such as those presented in the previous articles of this chapter, assume that the signal of interest has a constant amplitude. While this is a valid assumption in a wide range of scenarios, there are several important applications in which this assumption does not hold. Indeed, in many situations the signal may be subjected to a *random* amplitude modulation which behaves as multiplicative noise. Examples include fading in wireless communications [1], fluctuating targets in radar [2], and structural vibration of a spacecraft during launch and atmospheric turbulence [3]. In this article, we focus on non-parametric methods. In particular, we show that the Wigner-Ville distribution (defined in Section 2.1.4) is able to display the IF of a signal affected by multiplicative noise, and that this representation is optimal in the sense of maximum energy concentration for a linear FM signal. For higher-order polynomial FM signals, the use of the polynomial Wigner-Ville distribution (PWVD), presented in Article 5.4, is shown to give optimal representations. Statistical performance of each case will be presented here.

10.5.2 Linear FM Signal

In this section, we study the case of a linear FM signal and assume that the multiplicative noise is a *real-valued* process.

10.5.2.1 Optimality of the Wigner-Ville Spectrum

First we show that the Wigner-Ville spectrum (WVS) is optimal, in the sense of IF localization, for the time-frequency analysis of linear FM signals affected by multiplicative noise. Consider the signal $y(t)$ given by

$$y(t) = a(t) \cdot z(t) \tag{10.5.1}$$

where $a(t)$ is a non-zero-mean real-valued stationary noise and $z(t)$ is a deterministic FM signal given by $z(t) = \exp\{j\phi(t)\}$. For a linear FM signal, $\phi(t)$ is a second-order polynomial. Using the expectation operator notation, the autocorrelation of the signal above can be expressed as

$$
\begin{aligned}
\mathcal{K}_y(t, \tau) &= \mathrm{E}\left[y(t - \tfrac{\tau}{2})\, y^*(t + \tfrac{\tau}{2})\right] \\
&= \mathrm{E}\left[a(t - \tfrac{\tau}{2})\, a(t + \tfrac{\tau}{2})\right] \cdot \left\{z(t - \tfrac{\tau}{2})\, z^*(t + \tfrac{\tau}{2})\right\} \\
&= \mathcal{R}_a(\tau) K_z(t, \tau).
\end{aligned}
\tag{10.5.2}
$$

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The WVS of $y(t)$, which is defined as the Fourier transform of $\mathcal{K}_y(t, \tau)$ [4], can be expressed as

$$
\begin{align}
\mathcal{W}_y(t, f) &= \mathcal{F}_{\tau \to f} \{\mathcal{K}_y(t, \tau)\} \tag{10.5.3} \\
&= \mathcal{F}_{\tau \to f} \left\{ z(t - \tfrac{\tau}{2}) \, z^*(t + \tfrac{\tau}{2}) \right\} *_f \mathcal{F}_{\tau \to f} \left\{ \mathrm{E}\left[a(t - \tfrac{\tau}{2}) \, a(t + \tfrac{\tau}{2}) \right] \right\} \tag{10.5.4} \\
&= W_z(t, f) *_f S_a(f) \tag{10.5.5}
\end{align}
$$

where $*_f$ is the convolution operation in the frequency space.

If we express the non-zero-mean random process $a(t)$ as $a(t) = \mu_a + a_0(t)$, where μ_a is a constant mean of $a(t)$ and $a_0(t)$ is a zero-mean noise with autocorrelation $R_{a_0}(\tau)$, we can re-write the WVS of $y(t)$ as

$$
\mathcal{W}_y(t, f) = \mu_a^2 W_z(t, f) + S_{a_0}(f) *_f W_z(t, f). \tag{10.5.6}
$$

For the case of a linear FM signal, the Wigner-Ville distribution (WVD) is given in Article 2.1 by [4]

$$
W_z(t, f) = \delta(f - f_i(t)), \tag{10.5.7}
$$

where $f_i(t)$ is the signal IF and δ is the Dirac delta function. In this case, we obtain

$$
\mathcal{W}_y(t, f) = \mu_a^2 \delta(f - f_i(t)) + S_{a_0}(f - f_i(t)). \tag{10.5.8}
$$

Note that Eq. (10.5.8) exhibits the presence of a spectral line at the frequency $f_i(t)$ for all time instants. This means that, theoretically, the WVS always localizes the IF of a linear FM signal. This makes it a powerful tool in the analysis of linear FM signals affected by multiplicative noise. Also note that when $\mu_a = 0$, the WVS will not exhibit a peak at the signal IF, indicating a breakdown of the WVS to analyze the noisy signal.

As an illustration, consider a unit-modulus linear FM signal sampled at 1 Hz, whose frequency range lies between 0.1 Hz and 0.4 Hz. The signal length (in samples) is chosen as $N = 511$. This signal is multiplied by a real-valued i.i.d. Gaussian noise with a standard deviation $\sigma_a = 1$ and a mean equal to 0 and 1, respectively. Fig. 10.5.1 displays the WVS (one realization) of the noisy signal for both values of the mean. As expected, the WVS for the zero-mean case cannot reveal the signal IF; however, it can do so for the other case. In this last case, the peak of the WVS can be used to estimate the IF of the signal. In what follows, we will evaluate the statistical performance of such an estimator.

10.5.2.2   Statistical Performance Evaluation

Here, for a more complete study, we consider the presence of additive noise as well as the multiplicative noise. The objective is to derive the asymptotic variance of the IF estimator, based on the peak of the WVS, for this case.

Let the discrete-time version of the noisy signal be

$$
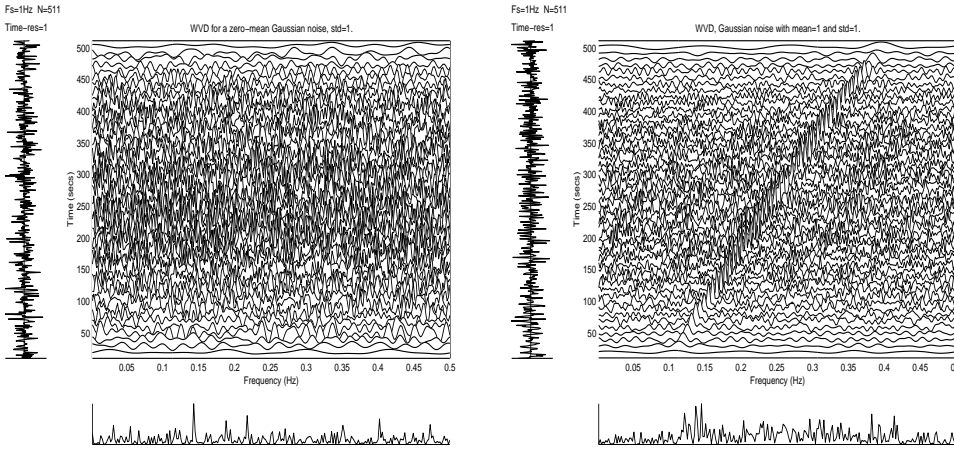y(n) = a(n)\, e^{j\phi(n)} + w(n) , \quad n = 0, \ldots, N - 1. \tag{10.5.9}
$$

Fig. 10.5.1: The Wigner-Ville spectrum of a linear FM signal affected by a real-valued Gaussian multiplicative noise. The noise variance is 1 (both plots) and its mean is 0 (left plot) and 1 (right plot).

The process $a(n)$ is considered to be a real-valued stationary Gaussian noise with mean and variance given by μ_a and σ_a^2, respectively. The complex zero-mean additive process $w(n)$ is assumed to be stationary, white, circular and Gaussian with variance equal to σ_w^2. In addition, both noises are assumed to be independent.

The WVS used to estimate the signal IF is defined, in the discrete-time domain, using the expression given in Article 6.1, as [4]

$$\mathcal{W}_z(n,f) = \mathrm{E}\left[2 \sum_{m=-M}^{M} y(n+m) \cdot y^*(n-m)\, e^{-j4\pi fm}\right]. \qquad (10.5.10)$$

Using straightforward derivations, we can show that, for increasing window length $(2M+1)$, the WVS converges in probability to $\mu_a^2 \delta(f - f_i(t))$ [5]. We can also show that the IF estimator asymptotic variance is approximately equal to [5]

$$\mathbf{Var}(\hat{f}_i(n)) = \frac{3}{(2\pi)^2 \mathcal{S}_w (2M+1)^3} \left(2 + \frac{2}{\mathcal{S}_a} + \frac{1}{\mathcal{S}_w}\right) \qquad (10.5.11)$$

where $\mathcal{S}_a = \mu_a^2 / \sigma_a^2$ and $\mathcal{S}_w = \mu_a^2 / \sigma_w^2$. Note that:

(i) When $\mu_a = 0$, the variance goes to infinity indicating that the WVS based estimator breaks down. This result confirms the analysis presented earlier.

(ii) When $\mu_a = A$ where A is a constant and $\sigma_a = 0$, i.e., the signal under consideration is just a constant amplitude linear FM signal embedded in noise, the variance expression can be-rewritten as

$$\mathbf{Var}(\hat{f}_i(n)) = \frac{3\sigma_w^2 [2A^2 + \sigma_w^2]}{(2\pi)^2 A^4 (2M+1)^3}.$$
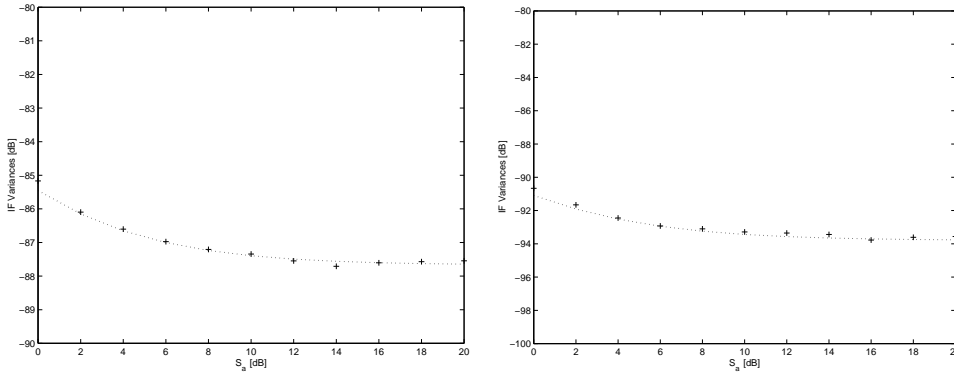
Fig. 10.5.2: Theoretical (dashed curve) and estimated ('+') variances of a linear FM signal corrupted by multiplicative and additive noise. Left plot is for $\mathcal{S}_w = 0\,\text{dB}$ and right plot is for $\mathcal{S}_w = 5\,\text{dB}$.

If, in addition, we assume high signal-to-noise ratio (SNR) (i.e., $A^2 \gg \sigma_w^2$) the asymptotic variance expression reduces to

$$\mathbf{Var}(\hat{f}_i(n)) = \frac{6\sigma_w^2}{(2\pi)^2(A^2)(2M+1)^3}$$

which is similar to the result obtained in [6].

The above theoretical results were confirmed by Monte-Carlo simulations. Specifically, we estimate the IF estimator variance using 5000 realizations of the signal given by (10.5.9). In Fig. 10.5.2, we display the theoretical (dashed curve) and the estimated ('+') variances plotted against $\mathcal{S}_a$ for $\mathcal{S}_w = 0$ and $5\,\text{dB}$, respectively.

If the signal under consideration is not a linear FM but a higher-order polynomial FM signal, the WVS becomes inappropriate because it introduces some artifacts which might hide the real features of the signal and its peak based IF estimator is biased for such signals [6]. In this situation, a different tool is needed for the analysis. This is the topic of the next section.

10.5.3   Polynomial FM Signals

Compared to the previous section, the extension here is two-fold: (i) the signal considered is assumed to be a polynomial FM signal of arbitrary order and (ii) the multiplicative noise is no longer limited to a real-valued process but is assumed to be a non-zero complex circular Gaussian process. Based on this, the noisy signal $y(t)$ is now written as

$$y(t) = a(t) \cdot z(t) + w(t) \tag{10.5.12}$$

where the stationary processes $a(t)$ and $w(t)$ are both assumed circular, complex, Gaussian and independent with means and variances given by (μ_a, σ_a^2) and $(0, \sigma_w^2)$,

respectively. The noiseless polynomial FM signal $z(t)$ is given by

$$z(t) = e^{j\phi(t)} = \exp\left\{ j\sum_{i=0}^{P} a_i t^i \right\} \qquad (10.5.13)$$

where the a_i are real coefficients and P is the order of the polynomial phase. Note that in the derivation below we do not require the knowledge of the coefficients a_i; we only assume a polynomial FM signal.

The IF of the signal $z(t)$ is given in Article 1.3 by

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = \sum_{i=1}^{P} i a_i t^{i-1}$$

and our primary objective here is to estimate $f_i(t)$ from the noisy signal $y(t)$. For that purpose, we use the polynomial Wigner-Ville distribution (PWVD) defined in Article 5.4 as

$$W_z^{(q)}(t, f) = \int_{-\infty}^{\infty} \prod_{i=1}^{q/2} z(t + c_i\tau) z^*(t + c_{-i}\tau) e^{-j2\pi f\tau} d\tau \qquad (10.5.14)$$

$$= \int_{-\infty}^{\infty} K_z^{(q)}(t, \tau) \cdot e^{-j2\pi f\tau} d\tau \qquad (10.5.15)$$

where q is an even integer which indicates the order of non-linearity of the PWVD. The coefficients c_i and c_{-i} $(i = 1, 2, \ldots, q/2)$ are calculated so that the PWVD is real and equal to

$$W_z^{(q)}(n, f) = \delta(f - f_i(t)),$$

for signals given by Eq. (10.5.13). Note that the realness of the PWVD implies that $c_i = -c_{-i}$. Also note that the WVD is a member of the PWVDs class with parameters $q = 2$ and $c_1 = -c_{-1} = 0.5$. Full details of the design procedure may be found in [7] and Articles 5.4 and 5.5.

The choice of the PWVD stems from the fact that it yields a continuum of delta functions around the IF for a given polynomial FM signal. This property implies that the peak of the PWVD can be used as an IF estimator for polynomial FM in a noisy environment. In [8], the statistical performance of this estimator was evaluated for noisy signals described by Eq. (10.5.12). It shows that this estimator is unbiased and its asymptotic variance is approximately equal to [8]

$$\mathbf{Var}(\hat{f}_i(n)) = \frac{6 (\sigma_a^2 + \sigma_w^2) \sum_{i=1}^{n_1} k_i^2}{(2\pi)^2 |\mu_a|^2 (2M+1)^3}. \qquad (10.5.16)$$

In the above expression, n_1 represents the number of the different coefficients c_i in the PWVD kernel, while k_i (for $i = 1, \ldots, n_1$) represents the multiplicity of each of these coefficients c_i, and $(2M + 1)$ is the window length considered in the PWVD discrete-time implementation. Note that:

(i) When $\mu_a = 0$, the variance goes to infinity indicating that the PWVD based estimator breaks down.

(ii) When $\mu_a = A$ where A is a constant and $\sigma_a = 0$, i.e., the signal under consideration is just a constant amplitude polynomial FM signal embedded in complex Gaussian noise, the variance expression can be-rewritten as

$$\mathbf{Var}(\hat{f}_i) = \frac{6 \, \sigma_w^2 \sum_{i=1}^{n_1} k_i^2}{(2\pi)^2 A^2 (2M+1)^3}.$$

The above expression is exactly similar to the result obtained in [6], which treats *constant* amplitude polynomial FM signals only.

10.5.3.1   Monte-Carlo Simulations

To confirm the validity of the above theoretical results, we consider the IF estimation of a quadratic FM signal at the middle of the signal interval. The peak of the sixth-order PWVD, whose signal kernel is given by [7]

$$\begin{aligned}
K_y^{(6)}(t,\tau) &= [y(t+0.62\tau) \, y^*(t-0.62\tau)] \cdot [y(t+0.75\tau) \, y^*(t-0.75\tau)] \\
&\times [y(t-0.87\tau) \, y^*(t+0.87\tau)],
\end{aligned}$$

is used here as the IF estimator. The noisy signal $y(t)$ is generated as suggested by Eq. (10.5.12). For this example, we choose the sampling period equal to $T = 1$, the signal length N equal to 129, the window length equal to $(2M + 1 = N = 129)$ and the noise variances to be equal (i.e., $\sigma_a^2 = \sigma_w^2$). In the simulations, the overall signal-to-noise ratio, defined as

$$\mathrm{SNR}_{w_1} = 10 \log_{10}(|\mu_a|^2/(\sigma_a^2 + \sigma_w^2)),$$

is varied in a 1 dB step from 0 to 15 dB. Monte-Carlo simulations for 1000 realizations are run for each value of SNR_{w_1}. The results of two different experiments, one performed for $|\mu_a| = 0.01$ and the other performed for $|\mu_a| = 1$, are displayed in Fig. 10.5.3 (left plot). We observe that, above a certain threshold, the estimated variances represented by '+' (for $|\mu_a| = 0.01$) and 'o' (for $|\mu_a| = 1$) are in total agreement with the theoretical ones given by Eq. (10.5.16) and represented by the continuous lines (superimposed).

   Simulations run under the same noise conditions for other polynomial FM signals using the appropriate PWVD order also confirm the theoretical results presented above. One such case is when $y(t)$ is a linear FM signal and the PWVD considered is the second-order PWVD. The results of this experiment are displayed in the right plot of Fig. 10.5.3.

10.5.4   Time-Varying Higher-Order Spectra

Time-Varying Higher-Order Spectra (TV-HOS) based on the polynomial Wigner-Ville distribution are defined as the expected value of the PWVD [9], namely

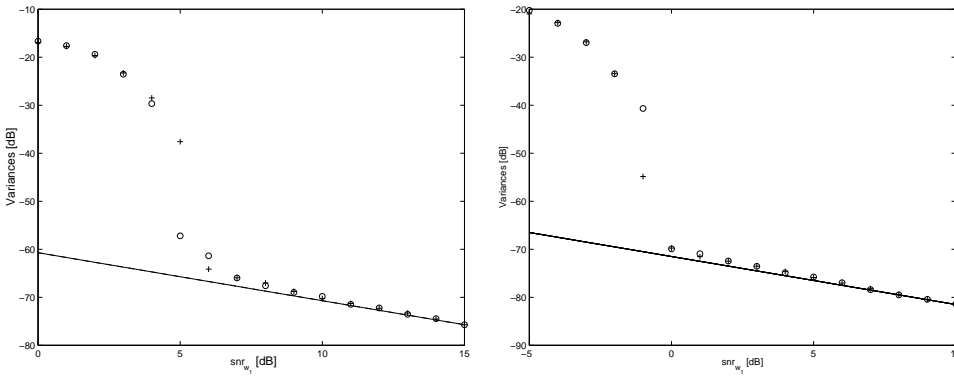$$\mathcal{W}_z^{(q)}(t,f) = \mathrm{E}\left[W_z^{(q)}(t,f)\right] \qquad (10.5.17)$$

Fig. 10.5.3: Results of the experiments for a **quadratic** (left plot) and a **linear** (right plot) FM signal corrupted by complex Gaussian multiplicative and additive noise processes. The continuous lines (superimposed) represent the theoretical variances while '+' and 'o' correspond to the estimated variances for $|\mu_a| = 0.01$ and $|\mu_a| = 1$, respectively.

where $W_z^{(q)}(t, f)$ is the qth-order PWVD defined by (10.5.14). Interchanging the expectation operator with the integration in the PWVD, one obtains

$$\mathcal{W}_z^{(q)}(t, f) = \int_{-\infty}^{\infty} \mathrm{E}\left[\prod_{i=1}^{q/2} z(t + c_i\tau)z^*(t + c_{-i}\tau)\right] e^{-j2\pi f\tau}d\tau \quad (10.5.18)$$

$$= \int_{-\infty}^{\infty} \mathcal{K}_z^{(q)}(t, \tau) \cdot e^{-j2\pi f\tau}d\tau \quad (10.5.19)$$

where $\mathcal{K}_z^{(q)}(t, \tau)$ represents a slice of a time-varying q^{th}-order moment function [9]. If the quantity $\mathcal{K}_z^{(q)}(t, \tau)$ is absolutely integrable, then, $\mathcal{W}_z^{(q)}(t, f)$ can be interpreted as a form of the time-varying higher-order moment spectrum. In [9], the authors showed that TV-HOS combine the advantages of classical time-frequency analysis with the benefits of higher-order spectra. To avoid the problem of non-superposition of the higher-order moments, the authors in [9] used higher-order cumulants instead.

It is important to note that since non-stationary random signals are non-ergodic, the ensemble averaging above cannot be replaced by time averaging. In this situation, local ergodicity has to be assumed.

Readers interested in TV-HOS are referred to [9] for more details, including examples of the efficacy of TV-HOS in the analysis of random FM signals affected by multiplicative noise. (See also Section 14.5.4 and the references in [9].)

10.5.5 Summary and Conclusions

The Wigner-Ville spectrum (WVS) and polynomial Wigner-Ville distributions (PWVDs) are considered for the analysis of polynomial FM signals corrupted by multiplicative and additive noise. In the noisy linear FM case, the WVS is shown

to give optimal IF localization. Accordingly, the peak of the WVS is proposed as an IF estimator. A statistical performance test shows that this estimator is very accurate even at low signal-to-noise ratio values. For the case of the noisy higher-order polynomial FM signal, the peak of the PWVD is shown to be a very consistent and accurate IF estimator.

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